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FINITE COMPLEXES WITH INFINITELY-GENERATED GROUPS OF SELF-EQUIVALENCES

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THE SET of (based) homotopy classes of homotopy equivalences from a (pointed) space X to itself forms a group $G(X)$, which is analogous to the group of automorphisms of a discrete group. It is of interest for many reasons and has attracted the attention of many authors. Recently, Sullivan[9] and Wilkerson[11] have shown that if X is a simply-connected finite CW-complex, then $G(X)$ is finitely-presented. Other authors had considered the structure of this group for various broad classes of spaces ([0], [3], [4], [10]). In specific cases (see, for example [7], [8]), explicit computations show that $G(X)$ is related to the group of units in a known algebra. In all these examples $G(X)$ is finitely-presented.

In this note, we give an example of a finite CW-complex X with infinitely-generated group of self-equivalences. We show that if $X = S^1 \vee S^2 \vee S^3$ (more generally, $S^1 \vee S^p \vee S^{2p-1}$, $p > 1$), then $G(X)$ is infinitely-generated. Along the way, we need to analyze the group of self-equivalences of $S^1 \vee S^2$ (more generally, $S^1 \vee S^p$, $p > 1$), which is in fact finitely-presented. Thus our example is perhaps as simple as possible.

Instead of considering spaces with few cells, another approach to finding a reasonable space X, with $G(X)$ infinitely-generated, would be to consider spaces with a finite number of non-zero homotopy groups, each homotopy group finitely-generated and $\pi_1(X)$ finitely-presented. In fact, there is an example of a finitely-presented group π for which $\mathrm{Aut}(\pi)$, the group of automorphisms of π, is not finitely-generated (see [5]). Hence if $X = K(\pi, 1)$, then $G(X)$ is not finitely-generated. However, $K(\pi, 1)$ is a rather large complex, and there is no reason to believe that there is a small subcomplex Y of $K(\pi, 1)$ with $G(Y)$ infinitely-generated. Thus these two approaches are different.

We work in the category of connected complexes with base-point, and base-point preserving maps and homotopies. When not necessary, the base-point shall be omitted from the notation.

PROPOSITION. *Let $X = S^1 \vee S^2$, and $G_0(X) \subsetneq G(X)$ be the subgroup consisting of those classes of self-equivalences which induce the identity on the homology of X. Then, we have*

(1) $G_0(X)$ is isomorphic to the integers. If γ: $S^2 \to S^1 \vee S^2$ and ι: $S^1 \to S^1 \vee S^2$ are the inclusions, then the integer n corresponds to a self-equivalence ϕ, $\{\phi\} \in G_0(X)$, such that $\phi\gamma = (n\iota) \cdot \gamma$, the latter being the action of $(n\iota)$ on γ.

(2) Let 1 be the identity, -1 a map of degree -1 on S^1 or S^2. Then $G(X)$ is generated by $-1 \vee 1$: $S^1 \vee S^2 \to S^1 \vee S^2$, $1 \vee -1$, and a generator of $G_0(X)$. (The relations on these generators are easily determined).

Proof. Define a function

$$\Phi: Z \to G_0(X)$$

by $\Phi(n) = \{\iota \vee ((n\iota) \cdot \gamma)\}$, which is easily seen to be a homomorphism. Clearly, $\Phi(n)$ induces an isomorphism on homotopy groups and the identity on homology groups. But if $n \neq m$, $\Phi(n)$ and $\Phi(m)$ induce different shift automorphisms on $\pi_2(X)$, so that Φ is a monomorphism.

In order to see that Φ is onto, we construct an injection of sets

$$\psi: G_0(X) \to \pi_2(X)$$

by $\psi(\{f\}) = \{f\gamma\}$, γ being the inclusion γ: $S^2 \to S^1 \vee S^2$. We set $\gamma_n = \{(n\iota) \cdot \gamma\}$ (the action of the fundamental group), so that $\pi_2(X)$ is the free abelian group generated by the γ_n. Note $\gamma_0 = \{\gamma\}$.

If we are given $\{f\}, \{g\} \in G_0(X)$, we write

$$\psi(\{f\}) = \sum_i a_i \gamma_i$$

and

$$\gamma(\{g\}) = \sum_i b_i \gamma_i,$$

where all but a finite number of the integer coefficients are 0. We then calculate

$$\begin{aligned}\psi(\{g\}\cdot\{f\}) &= g_*\Big(\sum_n a_n\gamma_n\Big) = \sum_n a_n g_*(\gamma_n)\\ &= \sum_n a_n g_*((n\iota)\cdot\gamma_0) = \sum_n a_n(n\iota)\cdot g_*(\gamma_0)\\ &= \sum_n a_n(n\iota)\cdot\Big(\sum_m b_m((m\iota)\cdot\gamma_0)\Big)\\ &= \sum_{n,m} a_n b_m(n\iota)\cdot((m\iota)\cdot\gamma_0)\\ &= \sum_{n,m} a_n b_m((n+m)\iota\cdot\gamma_0) = \sum_k\Big(\sum_{n+m=k} a_n b_m\Big)\gamma_k.\end{aligned}$$

We claim that if f is a homotopy equivalence, $\{f\}\in G_0(X)$, then all but one a_n are zero, and that non-zero a_n is $+1$. To see this, suppose g is a homotopy inverse for f, and more than one a_n is different from zero. Let $r<s$, with a_r and a_s being the a_n's of smallest and largest index which are non-zero. Similarly, let $u \leqq v$, with b_u and b_v being the smallest and largest non-zero terms.

Because $\psi(\{1_X\}) = 1\cdot\gamma_0$, and $\psi(\{1_X\}) = \psi(\{gf\}) = \sum_k(\sum_{m+n=k} a_n b_m)\gamma_k$, we have

$$\left.\begin{aligned}&\sum_{m+n=0} a_n b_m = 1\\ \text{and if } k\neq 0,\quad &\sum_{m+n=k} a_n b_m = 0.\end{aligned}\right\}\qquad (1)$$

Setting $k = v+s$, we have

$$\sum_{m+n=k} a_n b_m = a_s b_v \neq 0.$$

Setting $k = u + r < v + s$,

$$\sum_{m+n=k} a_n b_m = a_r b_u \neq 0,$$

in contradiction to formulas (1) above. Thus it is impossible for $r<s$.

Since f induces the identity on homology, the single non-zero coefficient must be $+1$. It follows at once that Φ is epimorphic, proving our first assertion.

To prove the second claim, note that the self-equivalences $-1 \vee 1$ and $1 \vee -1$ are their own inverses, and that given any self-equivalence f, we may compose with $-1\vee 1$, or $1\vee -1$, or $(-1\vee 1)\cdot(1\vee -1)$, to get an element of $G_0(X)$. It is then clear that $-1\vee 1$, $1\vee -1$, and $\Phi(1)$ generate the group $G(X)$.

We may now prove

THEOREM. *If $X = S^1 \vee S^2 \vee S^3$, then $G(X)$ is infinitely-generated.*

Proof. Since $G_0(X)\subseteq G(X)$ as above has finite index, it suffices to show $G_0(X)$ is infinitely-generated.

Let $\delta_n = \{(n\iota)\delta\}$, where $\delta: S^3 \to S^1\vee S^2\vee S^3$ is the inclusion. Let $\gamma_{r,s}$ denote the Whitehead product $[\gamma_r,\gamma_s] = [(r\iota)\cdot\gamma_0, (s\iota)\cdot\gamma_0]$, with γ_0 as above. Clearly, any element of $\pi_3(X)$ may be written in the form

$$\sum_n a_n\delta_n + \sum_{r,s} a_{r,s}\gamma_{r,s},$$

where the coefficients a_n and $a_{r,s}(r\neq s)$ are integers, and $a_{r,r}$ is an integer or half-integer (recall that the Whitehead product of a generator of $\pi_2(S^2)$ with itself is twice the class of the Hopf map). Also, only a finite number of these coefficients are non-zero.

Let $\{f\}$ be an element of $G_0(X)$. In analogy with the above proposition, we define $\Delta(\{f\})\in\pi_3(X)$ by $\Delta(\{f\})=\{f\delta\}$. Then we may write

$$\Delta(\{f\})=\delta_n+\sum_{r,s}a_{r,s}\gamma_{r,s},$$

because the same argument used in the proof of the above proposition may be used to show that all but one a_n are zero, and that a_n is $+1$.

If g is another such homotopy equivalence, with

$$\Delta(\{g\})=\delta_m+\sum_{r,s}b_{r,s}\gamma_{r,s},$$

we may calculate

$$\Delta(\{g\}\{f\})=\{(gf)\delta\}=g_{\#}\Big(\delta_n+\sum_{r,s}a_{r,s}\gamma_{r,s}\Big)$$

$$=(n\iota)g_{\#}(\delta_0)+\sum_{r,s}a_{r,s}[(r\iota)g_{\#}(\gamma_0),(s\iota)g_{\#}(\gamma_0)].$$

By our proposition, for some p, $\{g(\gamma)\}=\{(p\iota)\cdot\gamma\}=\gamma_p$. But then

$$\Delta(\{g\}\{f\})=(n\iota)\cdot\Big(\delta_m+\sum_{r,s}b_{r,s}\gamma_{r,s}\Big)+\sum_{r,s}a_{r,s}[(r\iota)(p\iota)\cdot\gamma_0,(s\iota)(p\iota)\cdot\gamma_0]$$

$$=\delta_{n+m}+\sum_{r,s}b_{r,s}\gamma_{r+n,s+n}+\sum_{r,s}a_{r,s}(\gamma_{r+p,s+p})$$

$$=\delta_{n+m}+\sum_{r,s}(b_{r-n,s-n}+a_{r-p,s-p})\gamma_{r,s}.$$

Now, given $\{f\}\in G_0(X)$, we define $N(\{f\})$ to be the largest non-negative integer d such that some $a_{t,t\pm d}$ is non-zero (recall that only a finite number of the $a_{r,s}$ coefficients are non-zero).

Clearly, $N(\{g\}\cdot\{f\})\leq\max(N(\{g\}),N(\{f\}))$. If $G_0(X)$ were finitely-generated, there would be an integer M with $N(\{f\})\leq M$, for all $\{f\}\in G_0(X)$. We shall show that $G_0(X)$ is infinitely-generated by constructing a self-equivalence f with $N(\{f\})$ being an arbitrary positive integer. To thus end, let $f|S^1\vee S^2$ be the inclusion and $f|S^3$ represent the homotopy class

$$\delta_0+\gamma_{N,0}$$

for an arbitrary positive integer N. To prove that f is an equivalence, define $g|S^1\vee S^2$ to be the inclusion, and let $g|S^3$ represent

$$\delta_0-\gamma_{N,0}.$$

Then $gf|S^1\vee S^2$ is the identity from $S^1\vee S^2$ to itself, while

$$\{gf|S^3\}=\Delta(\{g\}\{f\})=\delta_0+(-1+1)\gamma_{N,0}=\delta_0.$$

Therefore, $gf\simeq 1$, and similarly, $fg\simeq 1$, completing our proof.

We may now make various remarks.

(1) $G(S^1\vee S^2\vee S^3)$ is infinitely-generated because the Whitehead product elements $\gamma_{r,s}$ decompose into an infinite number of orbits under the action of π_1. • •

Similarly, $G(S^1\vee S^p\vee S^{2p-1})$, $p>1$, is infinitely generated.

On the other hand, $G(S^1\vee S^p\vee S^q)$, $p>1$, $p<q<2p-1$, is finitely generated. The generators for G_0 are given by the shift transformations for $S^1\vee S^p$ or $S^1\vee S^q$, as in our proposition, plus those self-equivalences f, for which $f|S^1\vee S^p$ is the identity, and $f|S^q$ is the sum of the inclusion of S^q and an additive generator of $\pi_q(S^p)$.

(2) In the following examples, the group of self-equivalences is finitely-generated and explicitly known.

(a) $G(S^n) = Z_2$

(b) $G(S^1 \vee \cdots \vee S^1)$ is the group of automorphisms of a free group. Explicit generators may be found on p. 111 of [2].

(c) $G(S^n \vee \cdots \vee S^n)$, $n > 1$, is a group of unimodular matrices. This is well-known to be finitely-generated. For a modern treatment, see [1].

(d) $G(RP(2)) = Z_2$. This follows from the fact that there are two antipode true homeomorphisms of S^2 which induce distinct homeomorphisms on $\pi_2(S^2)$.

(e) If T is an n-holed torus, $G(T)$ is the group of automorphisms of $\pi_1(T)$, which is a group with $2n$ generators and one relation. For the group of automorphisms of a group presented with 1 relation, see [6].

(3) This work and the known examples suggest the need to study the coherence problem for $G(X)$:

If $G(X)$ is finitely-generated, is $G(X)$ finitely-presented?

While this may be difficult for arbitrary arcwise connected spaces, for connected, finite complexes, it should be within reach of present techniques.

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